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# On geometric transitions in string compactifications

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## Abstract

We reconsider the study of the geometric transitions and brane/flux dualities in various dimensions. First we give toric interpretations of the topology changing transitions in the Calabi–Yau conifold and the  $Spin(7)$  manifold. The latter, for instance, can be viewed as three intersecting Calabi–Yau conifolds according to the  $\mathbb{CP}^2$  toric graph. Orbifolds of such geometries are given in terms of del Pezzo complex surfaces. Second we propose a four-dimensional F-theory interpretation of type IIB geometric transitions on the Calabi–Yau conifold. This gives a dual description of the M-theory flop in terms of toric mirror symmetry. In two dimensions, we study the geometric transition in a singular  $Spin(7)$  manifold constructed as a cone on  $SU(3)/U(1)$ . In particular, we discuss brane/flux duality in such a compactification in both type IIA and type IIB superstrings. These examples preserve one supercharge and so have  $\mathcal{N} = 1/2$  supersymmetry in two dimensions. Then, an interpretation in terms of F-theory is given.

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## 1. Introduction

It has been known for a long time that duality plays an interesting role in the context of string theory. More recently, geometric transitions have become a tool in understanding large  $N$  dualities between  $SU(N)$  gauge theory and closed string models [1–3]. Some well-known examples are provided by D-branes wrapped on cycles in manifolds with non-trivial holonomy groups, where a good description is given by the small coupling limit of the corresponding worldvolume gauge theory. After the geometric transition, the D-branes disappear and they are replaced by fluxes through cycles in the dual geometry, providing the adequate description of the same physics at strong coupling [1, 2]. In Calabi–Yau geometry, for instance, the

large- $N$  duality between D6-branes wrapped around the  $S^3$  of the deformed conifold,  $T^*S^3$ , and type IIA superstring on the small resolution of the conifold,  $O(-1) + O(-1)$  bundle over  $\mathbb{CP}^1$ , with fluxes has been studied in [2]. This result has been ‘lifted’ to M-theory [4] where it corresponds to the so-called flop duality in M-theory compactified on a manifold with  $G_2$  holonomy, for short, a  $G_2$  manifold. The mirror version of type IIA duality becomes the large- $N$  equivalence between D5-branes wrapped on the  $\mathbb{CP}^1$  of the resolved conifold and type IIB superstring with 3-form fluxes through the  $S^3$  of  $T^*S^3$ . This has been studied and extended to more general Calabi–Yau geometries [5–11].

Recently, similar studies have been done in three and two dimensions using respectively type IIA superstring compactified on a  $G_2$  manifold [12, 13], and type IIA(or B) on an eight-dimensional manifold with  $Spin(7)$  holonomy group, for short,  $Spin(7)$  manifold [3, 13]. This  $Spin(7)$  manifold is constructed as a cone on  $SU(3)/U(1)$  [14] and has a geometric transition involving a collapsing  $S^5$  and a growing  $\mathbb{CP}^2$ . In type II superstrings, this could be interpreted as a transition between two phases described by wrapped D-branes or R-R fluxes [3].

The aim of this work is to reconsider the study of the geometric transitions and D-brane/flux dualities in various dimensions. First we give toric interpretations of the topology changing transitions in the Calabi–Yau conifold and the  $Spin(7)$  manifold. The latter, for instance, can be viewed as three intersecting Calabi–Yau conifolds according to the  $\mathbb{CP}^2$  toric graph. Orbifolds of such geometries are given in terms of del Pezzo complex surfaces. Second we propose a four-dimensional F-theory interpretation of type IIB geometric transitions on the Calabi–Yau conifold. This gives a dual description of the M-theory flop in terms of toric mirror symmetry. In two dimensions, we study the geometric transition in a singular  $Spin(7)$  manifold constructed as a cone on  $SU(3)/U(1)$ . In particular, we discuss brane/flux duality in such a compactification in both type IIA and type IIB superstrings. Then, we engineer gauge symmetries going beyond the models given in [3]. These examples preserve one supercharge and so have  $\mathcal{N} = 1/2$  supersymmetry in two dimensions.

The organization of this work is as follows. In section 2, we briefly review the main line of the toric geometry method for treating manifolds with tori fibrations. In section 3, we give a toric interpretation of the topology changing transitions in such manifolds. In particular, we discuss the Calabi–Yau conifold transition, the  $Spin(7)$  manifold and its generalization to non-trivial geometries. In section 4, we reconsider the study of the transition in four dimensions by giving a new F-theory interpretation of the transition duality in type IIB superstring theory compactified on the Calabi–Yau conifold. In the case of type II superstrings on the  $Spin(7)$  manifolds, we give a conjecture on D-brane/flux duality in type IIA and type IIB with  $\mathcal{N} = 1/2$  in two dimensions. This is given in section 5. We end this study with discussion and some open questions.

## 2. Toric geometry

### 2.1. Projective spaces and odd-dimensional spheres

In this section, we collect a few facts on toric realizations of non-trivial geometries. These facts are needed later on when discussing the topology changing in manifolds with non-trivial holonomy groups. Roughly speaking, toric manifolds are, in general, complex  $n$ -dimensional manifolds with  $T^n$  fibration over real  $n$ -dimensional base spaces with boundary [15–17]. They exhibit toric actions  $U(1)^n$  allowing us to encode the geometric properties of the complex spaces in terms of simple combinatorial data of polytopes  $\Delta_n$  of the  $R^n$  space. In this correspondence, fixed points of the toric actions  $U(1)^n$  are associated with the vertices of the

polytope  $\Delta_n$ , the edges are fixed one-dimensional lines of a subgroup  $U(1)^{n-1}$  of the toric action  $U(1)^n$ , and so on.

To illustrate the main idea of toric geometry, let us describe the philosophy of this subject through certain useful examples, and then we give some generalizations useful later on.

- (i)  $\mathbb{CP}^1$  projective space. The simplest example, in toric geometry, is probably  $\mathbb{CP}^1$ . This manifold plays a crucial role in the building blocks of higher-dimensional toric varieties and in the study of the small resolution of singularities of local Calabi–Yau manifolds. Roughly,  $\mathbb{CP}^1$  has an  $U(1)$  toric action

$$z \rightarrow e^{i\theta} z \tag{1}$$

with two fixed points  $v_1$  and  $v_2$  on the real line. The latter points, which can be generally chosen as  $v_1 = -1$  and  $v_2 = 1$ , describe respectively north and south poles of the real two sphere  $S^2 \sim \mathbb{CP}^1$ . The corresponding one-dimensional polytope is just a segment  $[v_1, v_2]$  joining the two points  $v_1$  and  $v_2$ . Thus,  $\mathbb{CP}^1$  can be viewed as the segment  $[v_1, v_2]$  with a circle on top, where the circle vanishes at the end points  $v_1$  and  $v_2$ .

$$v_1 \text{ ————— } v_2 \tag{2}$$

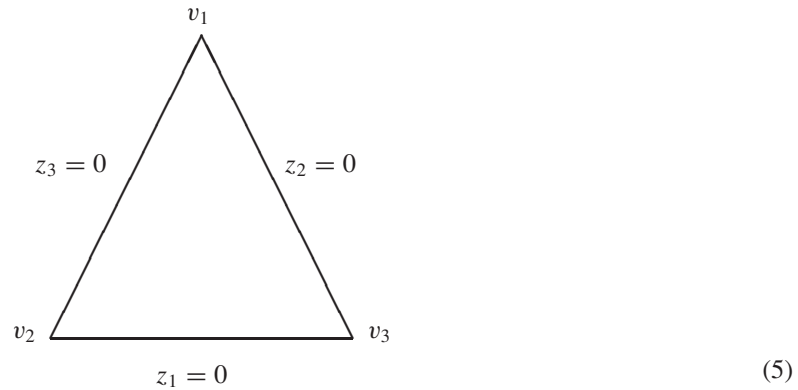
- (ii)  $\mathbb{CP}^2$  projective space. This is a complex two-dimensional toric variety defined by

$$\mathbb{CP}^2 = \frac{C^3 \setminus \{(0, 0, 0)\}}{C^*}, \tag{3}$$

where  $C^* : (z_1, z_2, z_3) \rightarrow (\lambda z_1, \lambda z_2, \lambda z_3)$ . This manifold admits an  $U(1)^2$  toric symmetry acting as follows,

$$(z_1, z_2, z_3) \rightarrow (e^{i\theta_1} z_1, e^{i\theta_2} z_2, z_3) \tag{4}$$

and exhibiting three fixed points  $v_1, v_2$  and  $v_3$ . The corresponding polytope  $\Delta_2$  is a finite sublattice of the  $\mathbf{Z}^2$  square lattice. It describes the intersection of three  $\mathbb{CP}^1$  defining a triangle  $(v_1 v_2 v_3)$  in the  $\mathbf{R}^2$  plane. Thus,  $\Delta_2$  has three edges, namely  $[v_1, v_2], [v_2, v_3]$  and  $[v_3, v_1]$ , stable under the three  $U(1)$  subgroups of  $U(1)^2$ . Two subgroups are just the two  $U(1)$  factors, while the third subgroup is the diagonal one.  $\mathbb{CP}^2$  can be viewed as a triangle over each point of which there is an elliptic curve  $T^2$ . This torus shrinks to a circle at each segment  $[v_i, v_j]$  and it shrinks to a point at each  $v_i$ . This is shown in the following figure:



This toric realization will play a crucial role in our present work. For  $n$ -dimensional projective spaces  $\mathbb{CP}^n$ , the toric representation is given by  $\mathbf{T}^n$  fibration over a  $n$ -dimensional simplex.

The above representation can be extended to some real manifolds, in particular the odd-dimensional real spheres being related to  $\mathbb{C}\mathbf{P}^n$  by

$$\mathbb{C}\mathbf{P}^n = \mathbf{S}^{2n+1}/\mathbf{S}^1. \tag{6}$$

In this way,  $\mathbf{S}^{2n+1}$  is a  $\mathbf{S}^1$  bundle over  $\mathbb{C}\mathbf{P}^n$ . Using this realization, one can give a toric representation for odd-dimensional spheres. Indeed, the 1-sphere, for example, is trivially realized as a  $\mathbf{T}^1 \sim \mathbf{S}^1$  over the zero-simplex—a point. As we have seen, the 3-sphere may be realized as a  $\mathbf{T}^2$  over a one-simplex—a line segment as the one in (2). This may be extended to the  $(2n + 1)$ -dimensional sphere  $\mathbf{S}^{2n+1}$  which may be described as a  $\mathbf{T}^{n+1}$  over an  $n$ -simplex. Of particular interest in this work is the 5-sphere  $\mathbf{S}^5$  being realized as the triangle (5) with a  $\mathbf{T}^3$  on top (whereas  $\mathbb{C}\mathbf{P}^2$  had a  $\mathbf{T}^2$  on top).

### 2.2. More general toric varieties

Let us now consider a more complicated example. The geometries that we will study in this subsection can also be described as quotient spaces. Consider the complex space  $C^{n+r}$  parametrized by  $z_1, \dots, z_{n+r}$  and  $rC^*$  actions given by

$$C^{*r} : z_i \rightarrow \lambda^{Q_i^\alpha} z_i, \quad i = 1, 2, \dots, n+r, \quad \alpha = 1, 2, \dots, r, \tag{7}$$

where  $Q_i^\alpha$  are integers. For each  $\alpha$  they form the so-called Mori vectors in toric geometry. They generalize the weight vector  $(w_i)$  of the complex  $n$ -dimensional weighted projective space  $\mathbf{WCP}_{w_1, \dots, w_{n+1}}^n$ . Now one can define a general toric variety  $\mathbf{V}^n$  by the following symplectic quotient space,

$$\mathbf{V}^n = \frac{C^{n+r} \setminus U}{C^{*r}}, \tag{8}$$

where  $U$  is a subset of  $C^k$  chosen by triangulation [16].

$\mathbf{V}^n$  has a  $T^n$  fibration, obtained by dividing  $T^{n+r}$  by the  $U(1)^r$  gauge symmetry

$$z_i \rightarrow e^{iQ_i^\alpha \vartheta^\alpha} z_i, \quad \alpha = 1, \dots, r, \tag{9}$$

where  $\vartheta^\alpha$  are the generators of the  $U(1)$  factors. It can be represented by a toric graph  $\Delta(\mathbf{V}^n)$  spanned by  $k = n + r$  vertices  $v_i$  in  $\mathbf{Z}^n$  lattice satisfying

$$\sum_{i=1}^{n+r} Q_i^\alpha v_i = 0, \quad \alpha = 1, \dots, r. \tag{10}$$

This geometric description of  $\mathbf{V}^n$  has a nice physical realization through the  $\mathcal{N} = 2$  linear sigma model. The theory has an  $U(1)^r$  gauge symmetry with  $n + r$  chiral fields  $\phi_i$  and a  $Q_i^\alpha$  matrix gauge charge [18]. In this way,  $\mathbf{V}^n$  is a solution of the vanishing condition of the D-term potential ( $D^\alpha = 0$ ), up to  $U(1)^r$  gauge transformations, namely

$$\sum_{i=1}^{n+r} Q_i^\alpha |\phi_i|^2 = \rho_\alpha, \tag{11}$$

where the  $\rho_\alpha$  are Fayet–Iliopoulos (FI) coupling parameters. The (local) Calabi–Yau condition is satisfied by

$$\sum_{i=1}^{n+r} Q_i^\alpha = 0, \quad \forall \alpha, \tag{12}$$

which means that the physical system flows in the IR to a non-trivial superconformal theory [18, 19].

Finally note that for a given toric complex manifold, one can construct its mirror using two-dimensional sigma model analysis [20, 21]. Indeed, the mirror version of the constraint equation (11), giving the superpotential in the Landau–Ginsburg (LG) models, reads

$$\sum_i a_i y_i = 0 \tag{13}$$

subject to

$$\prod_i y_i^{Q_i^\alpha} = 1. \tag{14}$$

In these equations,  $y_i$  are LG dual chiral fields which can be related, up to some field changes, to sigma model fields and where  $a_i$  can be identified with the complexified FI parameters, defining now the complex deformations of the LG Calabi–Yau superpotentials.

### 3. Toric varieties and geometric transitions

In this section, we reconsider the study of the topology changing of manifolds with exceptional holonomy group using toric geometry. In particular, we discuss the case of the Calabi–Yau manifold and  $Spin(7)$  manifolds.

#### 3.1. Toric Calabi–Yau conifold

Let us consider first the known example corresponding to three complex dimensions which is the so-called conifold. This manifold is defined by the following algebraic equation,

$$uv - xy = 0, \tag{15}$$

where the singularity is located at  $(u, v, x, y) = (0, 0, 0, 0)$ . There are basically two ways of smoothing this singularity, either by ‘toric’ small resolution or by complex deformation.

*Toric small resolution.* It consists in replacing the singular point  $(0, 0, 0, 0)$  by a toric  $\mathbb{CP}^1$  manifold. This resolution can be described by a  $\mathcal{N} = 2$  toric sigma model realization. The theory is an  $U(1)$  gauge model with four chiral matter fields  $\phi_i$  with the vector charge  $Q_i = (1, 1, -1, -1)$ . In this way, the conifold equation (15) can be solved in terms of the gauge invariant terms as follows:

$$u = \phi_1 \phi_3 \quad v = \phi_2 \phi_4 \quad x = \phi_1 \phi_4 \quad y = \phi_2 \phi_3. \tag{16}$$

Turning on the FI D-terms, which are given by

$$V_D = (|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 - \rho)^2, \tag{17}$$

corresponds to blowing up the origin by a toric  $\mathbb{CP}^1$ . Indeed, consider, for example, the case  $\rho > 0$ . Then the fields  $\phi_1$  and  $\phi_2$  cannot be zero, and these two coordinates define a  $\mathbb{CP}^1$  given (up to a  $U(1)$  gauge transformation) by

$$|\phi_1|^2 + |\phi_2|^2 = \rho. \tag{18}$$

The fields  $\phi_3$  and  $\phi_4$  can be regarded as non-compact coordinates parametrizing the normal directions for the fibres. Then the total space of the small resolution is a toric variety given by the bundle  $O(-1) + O(-1) \rightarrow \mathbb{CP}^1$ , which is topologically  $\mathbb{R}^4 \times \mathbb{S}^2$ . A similar analysis can be done for  $\rho < 0$  by exchanging the role of the base and the fibre. These two small resolutions ( $\rho < 0$  and  $\rho > 0$ ) of the conifold are related by the so-called flop transition.

*Complex deformation.* Besides the toric resolution we have discussed above, the conifold singularity can be deformed by keeping the Kahler structure and modifying the defining algebraic equation as follows,

$$uv - xy = \mu, \tag{19}$$

where  $\mu$  is a complex parameter. Now the singular point is replaced by an  $\mathbf{S}^3$  being obtained by taking a real parameter  $\mu$ ,  $u = \bar{v}$  and  $x = -\bar{y}$ . Up to changes of variables, one can show that the total geometry is nothing but  $T^*\mathbf{S}^3$ , whose topology is clearly  $\mathbb{R}^3 \times S^3$ . This is called the deformed conifold and is related to the resolved conifold by the so-called conifold transition.

The conifold transition admits a representation in toric geometry, where it can be understood as an enhancement or breaking, respectively, of the toric circle actions. On the one hand, the  $O(-1) + O(-1)$  bundle over  $\mathbb{C}\mathbf{P}^1$  has only one toric  $U(1)$  action, identified with the toric action on  $\mathbb{C}\mathbf{P}^1$  itself, while the deformed conifold  $T^*\mathbf{S}^3$  has a toric  $U(1)^2$  action since the spherical part can be viewed as a  $\mathbf{T}^2$  over a line segment. The torus is generated by the two  $U(1)$  actions

$$(u, v) \rightarrow (e^{i\theta_1} u, e^{-i\theta_1} v), \quad (x, y) \rightarrow (e^{i\theta_2} x, e^{-i\theta_2} y) \tag{20}$$

with  $\theta_i$  real. Thus, the blown-up  $\mathbf{S}^3$  may be described by the complex interval  $[0, \mu]$  with the two circles parametrized by  $\theta_i$  on top, where  $\mathbf{S}^1(\theta_1)$  collapses to a point at  $\mu$  while  $\mathbf{S}^1(\theta_2)$  collapses to a point at 0. The transition occurs when one of these circles refrains from collapsing while the other one collapses at both interval endpoints. This breaks the toric  $U(1)^2$  action to  $U(1)$ , and the missing  $U(1)$  symmetry has become a real line (over  $\mathbb{C}\mathbf{P}^1$ ). The resulting geometry is thus the resolved conifold.

### 3.2. Toric representation of the transition in the $Spin(7)$ manifold

In this subsection, we propose a picture for understanding the topology changing transition of the  $Spin(7)$  manifold discussed in [13] using toric geometry. The example we shall be interested in may be described as a singular real cone over the seven-dimensional Aloff–Wallach (coset) space  $SU(3)/U(1)$ . It was argued in [13] that there are two ways of blowing up the singularity, replacing the singularity by either  $\mathbb{C}\mathbf{P}^2$  or  $\mathbf{S}^5$ . The resulting smooth  $Spin(7)$  manifolds are given by  $\mathbb{R}^4 \times \mathbb{C}\mathbf{P}^2$  and  $\mathbb{R}^3 \times \mathbf{S}^5$  and are referred to as resolution and deformation, respectively, due to the similarity with the Calabi–Yau conifold ( $\mathbb{R}^4 \times \mathbb{C}\mathbf{P}^1$  and  $\mathbb{R}^3 \times \mathbf{S}^3$ ).

Here we shall re-consider the transition between these two manifolds in the framework of toric geometry. The basic idea [3] is to view the singular  $Spin(7)$  manifold (the real cone on  $SU(3)/U(1)$ ) as three intersecting Calabi–Yau conifolds associated with the triangular toric diagram (5). The deformed and resolved  $Spin(7)$  manifolds then correspond to the three intersecting Calabi–Yau conifolds being deformed or resolved, respectively.

Indeed, let us consider the complex linear space  $\mathbb{C}^3$  described by the three coordinates  $(z_1, z_2, z_3)$ . Let us introduce the constraint equation

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = r, \tag{21}$$

where  $r$  is real and positive. This defines a  $\mathbf{S}^5$  while the additional identification

$$(z_1, z_2, z_3) \sim (e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3) \tag{22}$$

(with  $\theta$  real) will turn it into a  $\mathbb{C}\mathbf{P}^2$ . In either case,  $r$  measures the size. With both conditions imposed, we can obtain the three resolved Calabi–Yau conifolds  $\mathbb{R}^4 \times \mathbb{C}\mathbf{P}^1$  ( $z_k = 0$ ),  $k = 1, 2, 3$  embedded in  $\mathbb{R}^4 \times \mathbb{C}^3$ , simply by setting one of the coordinates equal to 0. The resolution of the  $Spin(7)$  singularity reached by blowing up a  $\mathbb{C}\mathbf{P}^2$  can thus be described by three intersecting resolved conifolds over the triangle (5). Likewise, the deformation of the  $Spin(7)$  singularity constructed by blowing up a  $\mathbf{S}^5$  may be realized as three intersecting deformed Calabi–Yau conifolds  $\mathbb{R}^3 \times \mathbf{S}^3$  ( $z_k = 0$ ),  $k = 1, 2, 3$  over the same triangle. Note that this interpretation of the  $Spin(7)$  manifold as three intersecting Calabi–Yau manifolds over a triangle we have given corresponds to all three Calabi–Yau manifolds undergoing simultaneous conifold transitions.

### 3.3. More general geometric transitions

There are several generalizations of the above analysis of the  $Spin(7)$  manifold. We discuss some of them briefly here and leave others for future research.

*3.3.1. Orbifolds of  $Spin(7)$  manifolds and del Pezzo surfaces.* A simple generalization of the above idea is to consider orbifolds of  $Spin(7)$  manifolds. This involves del Pezzo complex surfaces  $dB_k$ ,  $k = 1, 2, \dots$  as base geometries. First we recall that  $dB_k$  are two-dimensional complex surfaces that are obtained by blowing up to  $k$  points in  $\mathbb{CP}^2$ . Alternatively  $dB_k$  can be obtained as Hirzebruch surfaces  $F_k$  being spheres fibred over spheres. In this way,  $dP_k$  can be identified with  $F_{k-1}$ . With the number of points restricted as  $k = 1, 2, 3$ , this defines the so-called toric del Pezzo surfaces. In toric geometry, the blowing up consists in replacing a point by  $\mathbb{CP}^1$  with a line segment as its toric diagram. The full del Pezzo surface will thus have a polygon with  $k + 3$  legs as its toric diagram. Let us now consider the resolved  $Spin(7)$  manifold and introduce a  $Z_2$  discrete group acting only on two homogeneous coordinates of  $\mathbb{CP}^2$  as follows:

$$(z_1, z_2, z_3) \rightarrow (-z_1, -z_2, z_3). \tag{23}$$

This transformation leads to a singular geometry. Locally, near such a singularity, this looks like an  $A_1$  ALE space. This is given by

$$uv = z^2 \tag{24}$$

where  $u = z_1^2$ ,  $v = z_2^2$  and  $z = z_1 z_2$ . The blow up of this singularity leads to  $\mathbb{CP}^1 \times \mathbb{CP}^1$  which can be identified with  $dP_1$ . So, the total geometry can be regarded now as four intersecting resolved conifolds according to a rectangle. After transition, we expect the same intersection but with the deformed Calabi–Yau conifolds. A similar analysis could be done for a generic value of  $k$ . In this way, the resulting geometry is given by  $k + 3$  intersecting Calabi–Yau manifolds according to a polygon.

*3.3.2. Higher-dimensional geometries.* Another possible generalization is to consider  $n$ -dimensional projective space  $\mathbb{CP}^n$ . Since  $S^{2n+1}$  can be described as a  $\mathbf{T}^{n+1}$  over an  $n$ -simplex it supports a toric  $U(1)^{n+1}$  action whereas  $\mathbb{CP}^n$  (which may be realized as a  $\mathbf{T}^n$  over an  $n$ -simplex) admits a toric  $U(1)^n$  symmetry. Like Calabi–Yau conifold, we are thus expecting that a geometric transition can take place, replacing a  $U(1)$  by the one-dimensional real line  $\mathbb{R}$ . Since the  $U(1)$  is associated with one of the  $S^1$  factors of  $\mathbf{T}^{n+1}$ , the transition essentially amounts to replacing  $\mathbf{T}^{n+1}$  by  $\mathbf{T}^n \times \mathbb{R}$ . Our interest is in real fibrations over the spaces  $S^{2n+1}$  and  $\mathbb{CP}^n$  so the relevant geometric transitions [3] may be read as

$$\text{(deformed geometry)} \quad \mathbb{R}^3 \times S^{2n+1} \longleftrightarrow \mathbb{R}^4 \times \mathbb{CP}^n \quad \text{(resolved geometry)}. \tag{25}$$

Relating it to toric geometry, this transition has a representation as intersecting conifolds over an  $n$ -simplex, provided by a simple combinatorial analysis. Indeed, the number of intersecting Calabi–Yau conifolds in the toric picture is equal to the number of one-dimensional edges of the simplex, namely  $\frac{1}{2}n(n + 1)$ . Similarly, one should also expect to be able to describe the transition in terms of intersecting  $Spin(7)$  manifolds over the  $n$ -simplex. In this case, the number of intersecting  $Spin(7)$  conifolds is equal to the number of two-dimensional faces of the simplex,  $\frac{1}{6}n(n^2 - 1)$ .



## 4. Geometric transitions in four dimensions

### 4.1. Duality in type IIA

Gopakumar and Vafa have recently conjectured that the  $SU(N)$  Chern–Simons theory on  $\mathbf{S}^3$  for large  $N$  is dual to topological strings on the resolved conifold [1]. In this way, the 't Hooft expansion of the Chern–Simons free energy has been shown to be in agreement, for all genera, with the topological string amplitudes on the resolved conifold. This duality has subsequently been embedded in type IIA superstring theory [2], where it was proposed that  $N$  D6-branes wrapped around  $\mathbf{S}^3$  of the deformed conifold is equivalent (for large  $N$ ) to type IIA superstring on the resolved conifold where the D6-branes replaced by  $N$  units of R-R 2-form fluxes through the 2-sphere ( $\mathbf{S}^2 \sim \mathbb{C}\mathbf{P}^1$ ) in the resolved conifold. This duality thus offers a way of understanding the same physics at strong coupling.

*M-theory interpretation.* The large- $N$  duality in type IIA superstring theory has also been lifted to M-theory on a  $G_2$  manifold [4, 12], where it is known to give the so-called flop duality. This gives an M-theory interpretation of the type IIA duality transition between the geometry involving branes and the one involving fluxes. Unlike the duality in string theory, the phase transition here is smooth and does not correspond to a topology changing geometric transition. To see this, consider M-theory on a seven-dimensional manifold  $X^7$  defined by

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = \rho, \quad (26)$$

where  $z_i$  are four complex variables and  $\rho$  is a real parameter. This equation defines an  $\mathbb{R}^4$  bundle over  $\mathbf{S}^3$  having  $G_2$  holonomy group. We now have two geometries related by a flop [4]. These are given by:

$$X_\rho^7 = \{(z_i) \in C^4 : |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = \rho\} \quad (27)$$

$$X_{-\rho}^7 = \{(z_i) \in C^4 : |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = -\rho\} \quad (28)$$

or, equivalently

$$X_\rho^7 = \mathbf{S}^3(z_1, z_2) \times C^2(z_3, z_4) \cong \mathbf{S}^3 \times R^4 \quad (29)$$

$$X_{-\rho}^7 = C^2(z_1, z_2) \times \mathbf{S}^3(z_3, z_4) \cong \mathbb{R}^4 \times \mathbf{S}^3. \quad (30)$$

In order to obtain, for instance, the small resolution with RR 2-form flux, we need to identify the M-theory circle with the circle of the Hopf bundle  $\mathbf{S}^1$  over  $\mathbf{S}^2$ , this being equivalent to identifying the '11th' circle with one of the toric geometry actions of the  $\mathbf{S}^3$  that we have discussed above. On the other hand, if we choose the M-theory circle in  $\mathbb{R}^4$  we get the deformed conifold geometry. Furthermore, this  $U(1)$  toric action turns out to have fixed points, giving rise to singularities that will correspond to D6-brane charges [15].

### 4.2. Duality in type IIB

The mirror version in type IIB superstring theory of type IIA duality states that the scenario with  $N$  D5-branes wrapped around the 2-sphere in the resolved conifold, is equivalent (for large  $N$ ) to 3-form fluxes through the  $\mathbf{S}^3$  of the deformed conifold. This has been generalized to other Calabi–Yau threefolds where the blown-up geometries involve several  $\mathbb{C}\mathbf{P}^1$  [5–11].

*F-theory interpretation.* So far we have reviewed how a type IIA duality (namely, the large- $N$  equivalence between a system of D6-branes wrapped on the finite  $\mathbf{S}^3$  of  $T^*\mathbf{S}^3$  and a type IIA background with 2-form flux on the  $\mathbf{S}^2$  of the resolved conifold and no branes) arises

from an M-theory flop [4]. A natural question is about the ‘mirror flop’, i.e. about an alternative description of the mirror version of this type IIA large- $N$  duality. Mirror symmetry between type IIA and type IIB compactifications maps the  $D6/S^3$  system to a system of D5-branes wrapped on the  $S^2$  of the resolved conifold and, correspondingly, maps the type IIA configuration with fluxes only (no branes) to a type IIB background with only 3-form fluxes on the blown-up  $S^3$  of the deformed conifold. Naively, the corresponding ‘mirror flop’ could be described by using mirror symmetry in M-theory compactifications. In this section we are going to see that what would be the ‘mirror flop’ is actually mirror symmetry in the base of a fourfold which is a (trivial) elliptic fibration over the conifold. Such a fourfold naturally describes an F-theory compactification [22]. To do so, we will proceed in two steps. First we note that the above-discussed conifold geometries are related by the local mirror symmetry transformation. To see this, consider the mirror geometry of the small resolution. Using (8)–(20), this is defined by solving the following constraint equations:

$$a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_4 = 0 \tag{31}$$

$$y_1 y_2 = y_3 y_4. \tag{32}$$

A solution of the last equation is given by  $y_1 = 1, y_2 = x, y_3 = y, y_4 = xy$ , and the mirror geometry becomes

$$a_1 + a_2 x + a_3 y + a_4 xy = 0. \tag{33}$$

At first sight, this geometry looks quite different from (19). However, one can relate it to the deformed conifold by taking the following limit in the complex moduli space:

$$a_1 = \mu, \quad a_2 = a_3 = 0, \quad a_4 = 1. \tag{34}$$

Now, we add the quadratic term. Note that this procedure has no influence on the physical moduli space. In this way, one can see that the mirror symmetry acts as follows,

$$\mathbb{R}^4 \times S^2 \longleftrightarrow \mathbb{R}^3 \times S^3 \tag{35}$$

and then it could be interpreted as a breaking/enhancement of an  $U(1)$  toric action, as discussed in the previous section.

The second step is to use this feature and our result in the context of the mirror duality between M-theory on  $G_2$  manifolds and F-theory on elliptic Calabi–Yau fourfolds [23]. Indeed, let us consider the conifold geometries described above as a mirror pair, and use the limit  $\mathbb{R}^4 = \mathbb{R}^3 \times S^1$  corresponding to the moduli space of one monopole in M-theory compactification. In such a limit, M-theory on (26) is dual to F-theory on a Calabi–Yau fourfold which is the product of  $T^2$  and the small resolution of the conifold. The type IIB meaning of this F-theory compactification has to be then the mirror system of the  $D6/S^3$  configuration, i.e. D5-branes wrapped on the  $S^2$  of the resolved conifold geometry. Now, if we apply toric mirror symmetry in the base of the fourfold on which we are compactifying F-theory, we are led to an F-theory compactification on  $T^2 \times T^*S^3$ . This provides the F-theory description of the other side of the type IIB transition (with only RR 3-form fluxes on the  $S^3$ ). This procedure immediately gives us another description of the M-theory flop of [4] in terms of mirror symmetry: if, as above, we look at (26) as an  $S^1$  fibre over the deformed conifold, we find that the flop can be alternatively described by mirror symmetry in the base of that fibration.

### 5. Geometric transition in two dimensions

In the previous section, we have presented the brane/flux duality in four dimensions which arises from the transition in the Calabi–Yau conifold. A natural question is whether a similar

description exists for the transition on the  $Spin(7)$  manifold compactification. In this section, we shall give such a description using the results of type II superstrings compactified on the conifold and the toric realization of the geometric transition of the  $Spin(7)$  manifolds. Following [3], we shall consider the large- $N$  limit two-dimensional gauge theories obtained by considering type II superstrings propagating on  $Spin(7)$  manifolds. The idea is to discuss the consequences of adding  $N$  wrapped D-branes to the set-up before letting the manifold undergo the geometric transition. In the transition from the resolved to the deformed  $Spin(7)$  manifold, we initially have D-branes wrapping  $\mathbb{C}\mathbb{P}^2$  (and its constituent 2-spheres). We conjecture that they are replaced, under the transition, by R-R fluxes through  $\mathbf{S}^5$  (and its constituent 3-spheres). Similarly in the transition from deformed to resolved  $Spin(7)$  manifolds, we conjecture that D-branes wrapped around  $\mathbf{S}^5$  (and its constituent 3-spheres) are replaced by R-R fluxes through  $\mathbb{C}\mathbb{P}^2$  (and its constituent 2-spheres). The kind of D-branes involved and the more detailed phase transition depend on which type II superstrings are propagating on  $Spin(7)$  manifolds. In what follows, we shall therefore consider type IIA and type IIB separately leading to different brane/flux dualities. First we consider type IIA geometry. Then we discuss the type IIB mode with several D-brane configurations involving D1, D3, D5 and D7-branes.

### 5.1. Duality in type IIA

First we consider type IIA superstring on the deformed  $Spin(7)$  manifold. A two-dimensional  $U(N)$  gauge theory can be obtained by wrapping  $N$  D6-branes around  $\mathbf{S}^5$ . The volume of  $\mathbf{S}^5$  described by  $r$  (21) is proportional to the inverse of the gauge coupling squared. This gauge model has only one supercharge. Thus we have  $\mathcal{N} = 1/2$ . At the transition point, the D6-branes disappear and are replaced by R-R 2-form fluxes through the 2-spheres embedded in  $\mathbb{C}\mathbb{P}^2$  in the resolved  $Spin(7)$  manifold.

*M-theory interpretation.* At this level, a natural question is about the analogue of the flop duality in four dimensions. The answer of this question may be given in terms of M-theory compactifications [3]. Indeed, consider a nine-dimensional manifold  $X_9$  with an  $U(1)$  isometry. M-theory compactified on  $X_9$  is then equivalent to type IIA superstring compactified on  $X_9/U(1)$ . We start with the resolved  $Spin(7)$  manifold and identify the extra eleventh compact dimension of M-theory with the  $\mathbf{S}^1$  that generates (22). In this way, the extra M-theory circle becomes the fibre in the definition of  $\mathbf{S}^5$  as a  $\mathbf{S}^1$  fibration over  $\mathbb{C}\mathbb{P}^2$ . We thus end up with an  $\mathbb{R}^4$  bundle over  $\mathbf{S}^5$  as the compactification space in M-theory. As a consequence, the moduli space of M-theory on such a background is parametrized by the real parameter  $r$  defining the volume of  $\mathbf{S}^5$  (21), and cannot be complexified by the C-field. Starting with the resolved  $Spin(7)$  manifold, on the other hand, the eleventh M-theory dimension is obtained by extending  $\mathbb{R}^3$  to  $\mathbb{R}^4$  with the isometry being a trivial  $U(1)$  action on the fibre  $\mathbb{R}^4$ . Using arguments similar to those in [4], it is conjectured [3] that this lift to M-theory gives rise to a (smooth) flop transition in the  $\mathbb{R}^4$  bundle over  $\mathbf{S}^5$  where a 5-sphere collapses and is replaced by a new 5-sphere. In our scenario, however, the physics resulting from the type IIA superstring compactification undergoes a singular phase transition due to the absence of 5-form gauge field in the spectrum.

### 5.2. Duality in type IIB

Now we consider type IIB superstring on the resolved  $Spin(7)$  manifold. In this case, two-dimensional gauge models can be engineered using several brane configurations. In particular, we have the following D-brane realizations:

- $N$  parallel D1-branes filling the two-dimensional spacetime (and placed at the singular point of the  $Spin(7)$  manifold). This gives an  $U(N)$  gauge symmetry in spacetime.
- $N$  parallel wrapped D3-branes over real 2-cycles and filling the two-dimensional spacetime (and placed at the singular point of the  $Spin(7)$  manifold). This leads also to  $2D U(N)$  theory.
- One may also have systems involving D3-branes and D1-branes. This may lead to  $U(N_1) \times U(N_2)$  product gauge symmetry.
- $N$  parallel wrapped D5-branes over  $\mathbb{C}\mathbf{P}^2$  and filling the two-dimensional spacetime (and placed at the singular point). This leads to  $U(N)$  gauge symmetry.
- $N$  parallel wrapped D7-branes over real 6-cycles in  $Spin(7)$  manifold and filling the two-dimensional spacetime (and placed at the singular point). This gives an  $U(N)$  gauge symmetry.
- One may also have systems involving D7-branes, D5-branes, D3-branes and D1-branes. This may lead to  $U(N_1) \times U(N_2) \times U(N_3) \times U(N_4)$  product gauge symmetry.

To realize two last configurations, we need a 6-cycle. To get that, we should mod out, the resolved  $Spin(7)$  manifold, by a  $Z_2$  discrete group acting on the fibre direction of  $\mathbb{R}^4$  bundle over  $\mathbb{C}\mathbf{P}^2$ . This corresponds to ALE space with  $A_1$  singularity over  $\mathbb{C}\mathbf{P}^2$ . After the deformation of this singularity, one gets a 6-cycle given by  $\mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^2$ . In shrinking limit, this will correspond to 6-cycle/5-cycle transition in superstring compactifications. This looks like the case of 4-cycle/3-cycle transition studied in [7]. In particular, after the transition, one may have  $\mathbf{S}^5/Z_2$  as a dual cycle.

Now return to the large- $N$  limit duality in type IIB. Here, we are not going to deal with the all above brane configurations. However, we consider a system with only D5-branes. Then we discuss a case where we have D3-branes. Since the type IIB superstring does not support 4-forms, one can wrap D5-branes around  $\mathbb{C}\mathbf{P}^2$ . As we have seen, a two-dimensional gauge model with  $U(N)$  gauge symmetry can be obtained by wrapping  $N$  D5-branes on  $\mathbb{C}\mathbf{P}^2$ . The volume of  $\mathbb{C}\mathbf{P}^2$  described by  $r$  (21) is proportional to the inverse of the gauge coupling squared. This two-dimensional model has only one supercharge so  $\mathcal{N} = 1/2$ . Now, when the manifold undergoes the geometric transition to the deformed  $Spin(7)$  manifold, the  $N$  D5-branes disappear and we expect a dual physics with  $N$  units of R-R three-fluxes through the compact 3-cycles,  $\mathbf{S}^3$ , in the intersecting Calabi–Yau threefolds. These fluxes could be accompanied by some NS-NS fluxes through the non-compact dual 3-cycles in the six-dimensional deformed conifolds. In order to handle the associated divergent integrals, one would have to introduce a cut-off to regulate the infinity [5].

So far, we have studied the geometric transition involving only D5-branes on the  $Spin(7)$  manifold. Now we would like to go beyond this model by adding, for instance, D3-branes. We will show that this brane configuration could be related to del Pezzo surfaces. Here, our construction of such brane gauge models will be based on  $\mathcal{N} = 2$  sigma model realization of the internal compact geometry. For simplicity, let us consider  $dP_1$ . Indeed, this can be realized by  $U_1(1) \times U_2(1)$  gauge theory with four chiral fields  $\phi_i$  with charges

$$Q_{(1)} = (1, 1, 1, 0) \quad Q_{(2)} = (0, 0, 1, 1).$$

The vanishing condition of D-terms reads as

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = \alpha_1 \quad |\phi_3|^2 + |\phi_4|^2 = \alpha_2$$

and defines  $dP_1$ . In what follows we will identify the  $U_1(1) \times U_2(1)$  sigma model gauge group with the gauge symmetry living on the worldvolume of D-branes. Indeed, omitting first the

$U_2(1)$  and after the wrapping procedure,  $U_1(1)$  gauge symmetry can be identified with an  $U(1)$  gauge field living on the D5-brane worldvolume<sup>1</sup>. However, due to the presence of  $U_2(1)$ , we really could introduce an extra brane. In this way,  $U_2(1)$  can be identified with the  $U(1)$  gauge field living on its worldvolume. Since  $U_2(1)$ , in  $\mathcal{N} = 2$  sigma model, corresponds to a blowing-up 2-cycle, we should add a D3-brane. The latter can wrap such a cycle and leads to an extra  $U(1)$  factor in two-dimensional spacetime. So, the general gauge symmetry is then  $U(1) \times U(1)$ . In what follows, we should note the following points:

1. In the vanishing limit of the 2-cycle, where  $dP_1$  reduces to  $\mathbb{C}\mathbb{P}^2$ , one gets only one  $U(1)$  factor corresponding to one D5-brane.
2. The above model could be generalized for the case where we have more than one brane. If we assume that one has the same number of D5 and D3-branes, then the gauge group should be  $U(N) \times U(N)$ .
3. After the transition point, these branes disappear and replaced by 3-form and 5-form fluxes on  $\mathbf{S}^5/Z_2$ .

We conclude this work with a comment regarding a F-theory interpretation of type IIB geometric transition in the compactification on the above  $Spin(7)$  manifold. This is given by a F-theory compactification on a ten-dimensional manifold involving a  $\mathbf{S}^5$  flop transition. While we were thinking on this question after completing the first version of this work, [24] appeared which has a considerable overlap with this comment. In [24], a 11-dimensional interpretation has been given. However, here we give a direct compactification on a ten-dimensional manifold preserving only one supercharge in two dimensions. Based on a close analogy to the Calabi–Yau case, we propose the following ten-dimensional,

$$T^*(\mathbf{S}^5)/\sigma, \tag{36}$$

where  $T^*(\mathbf{S}^5)$  is the complex deformation of the singular Calabi–Yau five-folds given by

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 = 0. \tag{37}$$

This manifold has nice features supporting our proposition:

- (1) In string theory compactifications,  $T^*(\mathbf{S}^5)$  preserves 1/16 of initial supercharges and in the presence of  $\sigma$  it should be 1/32. Thus, F-theory on the above ten-dimensional manifold leads to only one supercharge in two dimensions.
- (2) The presence of  $\sigma$  may lead to a  $\mathbf{S}^5$  flop transition in the quotient space. It is easy to see this by taking  $w_j$  as  $x_j + iy_j$  and rewriting the above algebraic equation. In this way,  $T^*(\mathbf{S}^5)$  can be described by

$$\mathbf{x} \cdot \mathbf{x} - \mathbf{y} \cdot \mathbf{y} = r, \quad \mathbf{x} \cdot \mathbf{y} = 0, \tag{38}$$

where  $r$  is a real parameter describing the size of  $\mathbf{S}^5$ . If we think that  $\sigma$  acts as  $x \rightarrow y$ , and  $r \rightarrow -r$ , then we have two spheres being connected by the so-called flop transition.

- (3) After a compactification, the reduced manifold may involve three Calab-Yau conifolds. To see this let us take a simple linear coordinate from  $(w_1, \dots, w_6)$  to  $(z_1, z_2, z_3, z_4, u, v)$ . This allows one to describe the above manifold by

$$\sum_{1 \leq i < j \leq 4} z_i z_j = z_4(z_3 + z_2 + z_1) + z_3(z_2 + z_1) + z_2 z_1 = uv \tag{39}$$

<sup>1</sup> This model can be identified with the one studied in the previous subsection.

such that

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{pmatrix} = \begin{pmatrix} \sqrt{3} & i & i & i & 0 & 0 \\ i & \sqrt{3} & i & i & 0 & 0 \\ i & i & \sqrt{3} & i & 0 & 0 \\ i & i & i & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & d & -id \\ 0 & 0 & 0 & 0 & -id & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ u \\ v \end{pmatrix} \tag{40}$$

with

$$d = \sqrt{\frac{2 + \sqrt{3}}{2}} + i\sqrt{\frac{2 - \sqrt{3}}{2}}. \tag{41}$$

Equation (39) describes six Calabi–Yau conifolds given by  $uv = z_i z_j; z_k = z_\ell = 0, i \neq j \neq k \neq \ell$ . However if we identify one of the  $z_i$  complex variables with the  $T^2$  torus of F-theory compactification, this number reduces to three and it is in agreement with the  $spin(7)$  geometry involving three Calabi–Yau manifolds.

- (4) Finally, the topology of  $spin(7)$  manifold can be obtained by identifying the  $T^2$  torus of F-theory with the toric actions of  $T^*(S^5)/\sigma$ . Using the results of section 2 and [17], one can get the resolved  $spin(7)$  by choosing one circle in the  $S^5$  base geometry and one circle in the corresponding cotangent directions, while the deformed  $spin(7)$  manifold is obtained by identifying the two F-theory circles with toric actions of the cotangent bundles.

### 6. Discussions

In this study, geometric transitions in type II superstrings on Calabi–Yau conifold and  $Spin(7)$  manifolds have been discussed using toric geometry. In the Calabi–Yau case, we have proposed a new F-theory interpretation for type IIB propagating on the conifold. Following [3] on  $Spin(7)$  manifolds, we have given a picture for understanding the topology changing transition in the  $Spin(7)$  manifolds in terms of the Calabi–Yau conifold transition. Then, we have studied brane/flux dualities in two dimensions using several brane configurations. This study gives first examples of geometric transitions with  $\mathcal{N} = 1/2$  supersymmetry. Then, an interpretation in terms of F-theory has been given.

This work opens up for further discussions. We shall collect some of them.

- In this study, we have considered the  $Spin(7)$  manifold as intersecting Calabi–Yau three-folds over a triangle where the  $Spin(7)$  transition corresponds to three simultaneous conifold transitions. It would be interesting to study the geometries associated with individual conifold transitions.
- It should be nice to find a sigma model explanation of the  $Spin(7)$  transition.
- We expect that all studies which have been done for the Calabi–Yau conifold could be pushed to  $Spin(7)$  manifolds. It would be nice to study the analogue of the Klebanov–Witten model on the conifold [25] for  $Spin(7)$  manifold.

We leave these questions for future works.

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